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Pseudoclassical Description of Massive Dirac Particles in Odd Dimensions

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A pseudoclassical model is proposed to describe massive Dirac (spin-one-half) particles in arbitrary odd dimensions. The quantization of the model reproduces the minimal quantum theory of spinning particles in such dimensions. A dimensional duality between the model proposed and the pseudoclassical description of Weyl particles in even dimensions is discussed.

1. INTRODUCTION

One can construct a pseudoclassical model to describe massive Dirac (spin-one-half) particles in 3 + 1 dimensions (Berezin and Marinov, 1975, 1977; Brink *et al.*, 1976, 1977; Casalbuoni, 1976; Barducci *et al.*, 1976; Balachandran *et al.*, 1977; Henneaux and Teitelboim, 1982; Gitman and Tyutin, 1990a; Fradkin and Gitman, 1991). Its generalization to the case of even dimensions $D = 2n$, $n = 3, 4, \ldots$, can be done by means of direct dimensional extension (Grigoryan and Grigoryan, 1991). The corresponding action has the form

$$S = \int_0^1 \left[-\frac{\dot{x}^2}{2e} - e\frac{m^2}{2} + i\left(\frac{\dot{x}_\mu \psi^\mu}{e} - m\psi^D\right)\chi - i\psi_a\dot{\psi}^a \right] d\tau \tag{1}$$

where $\dot{x}^2 = \dot{x}_\mu \dot{x}^\mu$; the Greek (Lorentz) indices $\mu, \nu, \ldots$ run over $0, 1, \ldots, D-1$, whereas the Latin ones a, b run over $0, 1, \ldots, D$;

$$\eta_{\mu\nu} = \mathrm{diag}(\underbrace{1, -1, \ldots, -1}_{D}), \qquad \eta_{ab} = \mathrm{diag}(\underbrace{1, -1, \ldots, -1}_{D+1})$$

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The variables x^μ and e are even and ψ^n, χ are odd. The quantization of the model leads to the Dirac quantum theory of the spin-one-half particle (to the Dirac equation).

Attempts to extend the pseudoclassical description to the arbitrary odd-dimensional case have met some problems, which are connected with the absence of an analog of the γ^5-matrix in such dimensions. For instance, in $2n + 1$ dimensions the direct generalization of the standard action (Berezin and Marinov, 1975, 1977; Brink *et al.*, 1976, 1977; Casalbuoni, 1976; Barducci *et al.*, 1976; Balachandran *et al.*, 1977; Henneaux and Teitelboim, 1982; Gitman and Tyutin, 1990a; Fradkin and Gitman, 1991) does not reproduce a minimal quantum theory of spinning particles, where particles with spin 1/2 and $-1/2$ have to be considered as different. In Plyuschchay (1993) and Cortes *et al.* (1993) modifications of the standard action were proposed to solve the problem. From our point of view both have essential shortcomings and the problem is not closed. For instance, the first action of Plyuschchay (1993) and Cortes *et al.* (1993) is classically equivalent to the standard action and does not provide the required quantum properties in the course of canonical and path-integral quantization. Moreover, it is P- and T-invariant, so that an anomaly is present. Another one in Plyuschchay (1993) and Cortes *et al.* (1993) does not obey supersymmetries and therefore loses the main attractive feature in such models, which allows one to treat them as prototypes of superstrings or some modes in superstring theory. Gitman *et al.* (1995a) proposed a new pseudoclassical model for a massive Dirac particle in 2 + 1 dimensions which obeys all the necessary symmetries, is P- and T-noninvariant, and reproduces the minimal quantum theory of the Dirac particle in 2 + 1 dimensions. It turns out to be possible to generalize this model to the arbitrary odd-dimensional case. We present such a generalization in the present paper. First, we consider the Hamiltonization of the theory and its quantization. Then we discuss a remarkable dimensional duality between the model proposed and the pseudoclassical description of massless spinning particles in even dimensions.

2. PSEUDOCLASSICAL DESCRIPTION

In odd dimension $D = 2n + 1$ we propose the following action to describe spinning particles:

$$S = \int_0^1 \left[-\frac{z^2}{2e} - e\frac{m^2}{2} - \iota m\psi^{2n+1}\chi - \frac{s}{2^n} m\kappa - \iota\psi_a\dot{\psi}^a \right] d\tau$$

$$\equiv \int_0^1 L\, d\tau, \qquad z^\mu = \dot{x}^\mu - \iota\psi^\mu\chi + \frac{(2\iota)^n}{(2n)!} \epsilon^{\mu\rho_1\cdots\rho_{2n}}\psi_{\rho_1}\cdots\psi_{\rho_{2n}}\kappa \qquad (2)$$

Here a new even variable κ is introduced and $\epsilon^{\mu\nu\ldots\lambda}$ is the Levi-Civita tensor density in $2n + 1$ dimensions normalized by $\epsilon^{01\ldots 2n} = 1$; s is an even constant of the Berezin algebra. We suppose that x^μ and ψ^μ are Lorentz vectors and e, κ, ψ^{2n+1}, χ are scalars, so that the action (2) is invariant under the restricted Lorentz transformations (but not P- and T-invariant). There are three types of gauge transformations under which the action (2) is invariant: reparametrizations

$$\delta x^\mu = \dot{x}^\mu \xi, \qquad \delta e = \frac{d}{d\tau}(e\xi), \qquad \delta\psi^a = \dot{\psi}^a \xi, \qquad \delta\chi = \frac{d}{d\tau}(\chi\xi),$$
$$\delta\kappa = \frac{d}{d\tau}(\kappa\xi) \tag{3}$$

with an even parameter $\xi(\tau)$; supertransformations

$$\delta x^\mu = \iota\psi^\mu\epsilon, \qquad \delta e = \iota\chi\epsilon, \qquad \delta\psi^\mu = \frac{z^\mu}{2e}\epsilon, \qquad \delta\psi^{2n+1} = \frac{m}{2}\epsilon,$$
$$\delta\chi = \dot{\epsilon}, \qquad \delta\kappa = 0 \tag{4}$$

with an odd parameter $\epsilon(\tau)$; and additional supertransformations

$$\delta x^\mu = -\frac{(2\iota)^n}{(2n)!}\epsilon^{\mu\rho_1\ldots\rho_{2n}}\psi_{\rho_1}\ldots\psi_{\rho_{2n}}\theta,$$
$$\delta\psi^\mu = -\frac{\iota}{e}\frac{(2i)^n}{(2n)!}\epsilon^{\mu\rho_1\ldots\rho_{2n}}z_{\rho_1}\psi_{\rho_2}\ldots\psi_{\rho_{2n}}\theta,$$
$$\delta\kappa = \dot{\theta} + \frac{s}{m}\frac{2^{2n+1}\iota^n(n-1)z_\mu}{(2n)!e}\epsilon^{\mu\rho_1\ldots\rho_{2n}}\dot{\psi}_{\rho_1}\psi_{\rho_2}\ldots\psi_{\rho_{2n}}\theta,$$
$$\delta e = \delta\psi^{2n+1} = \delta\chi = 0 \tag{5}$$

with an even parameter $\theta(\tau)$.

The total angular momentum tensor $M_{\mu\nu}$ is

$$M_{\mu\nu} = x_\mu\pi_\nu - x_\nu\pi_\mu + \iota[\psi_\mu, \psi_\nu] \tag{6}$$

where $\pi_\nu = \partial L/\partial\dot{x}^\nu$.

Going over to the Hamiltonian formulation, we introduce the canonical momenta

$$\pi_\mu = \frac{\partial L}{\partial\dot{x}^\mu} = -\frac{1}{e}z_\mu, \qquad P_e = \frac{\partial L}{\partial\dot{e}} = 0, \qquad P_\chi = \frac{\partial_r L}{\partial\dot{\chi}} = 0,$$
$$P_\kappa = \frac{\partial L}{\partial\dot{\kappa}} = 0, \qquad P_a = \frac{\partial_r L}{\partial\dot{\psi}^a} = -\iota\psi_a \tag{7}$$

It follows from (7) that there exist primary constraints

$$\Phi_1^{(1)} = P_e, \qquad \Phi_2^{(1)} = P_\chi, \qquad \Phi_3^{(1)} = P_\kappa, \qquad \Phi_{4a}^{(1)} = P_a + i\psi_a \tag{8}$$

Constructing the total Hamiltonian $H^{(1)}$ according to the standard procedure (Dirac, 1964; Gitman and Tyutin, 1990b), we get $H^{(1)} = H + \lambda_A \Phi_A^{(1)}$, where

$$H = -\frac{e}{2}(\pi^2 - m^2) + \iota(\pi_\mu \psi^\mu + m\psi^{2n+1})\chi$$
$$- \left[\frac{(2\iota)^n}{(2n)!}\epsilon^{\mu\rho_1\cdots\rho_{2n}}\pi_\mu\psi_{\rho_1}\dots\psi_{\rho_{2n}} - \frac{1}{2^n}sm\right]\kappa \tag{9}$$

From the consistency conditions $\dot{\Phi}^{(1)} = \{\Phi^{(1)}, H^{(1)}\} = 0$ we find secondary constraints $\Phi^{(2)} = 0$,

$$\Phi_1^{(2)} = \pi_\mu \psi^\mu + m\psi^{2n+1}, \qquad \Phi_2^{(2)} = \pi^2 - m^2,$$
$$\Phi_3^{(2)} = \frac{(2\iota)^n}{(2n)!}\epsilon^{\mu\rho_1\cdots\rho_{2n}}\pi_\mu\psi_{\rho_1}\dots\psi_{\rho_{2n}} - \frac{1}{2^n}sm \tag{10}$$

and determine λ, which corresponds to the primary constraints $\Phi_4^{(1)}$. No more secondary constraints arise from the consistency conditions, and the Lagrangian multipliers corresponding to the primary constraints $\Phi_i^{(1)}$, $i = 1, 2, 3$, remain undetermined. The Hamiltonian (9) is proportional to the constraints, as one would expect in the case of a reparametrization-invariant theory. One can go over from the initial set of constraints $\Phi^{(1)}$, $\Phi^{(2)}$ to the equivalent ones $\Phi^{(1)}$, $\tilde{\Phi}^{(2)}$, where $\tilde{\Phi}^{(2)} = \Phi^{(2)}(\psi \to \tilde{\psi} = \psi + \frac{1}{2}\iota\Phi_4^{(1)})$. The new set of constraints can be explicitly divided into a set of the first-class constraints, which are $(\Phi_i^{(1)}$, $i = 1, 2, 3$, $\tilde{\Phi}^{(2)})$, and a set of second-class constraints $\Phi_4^{(1)}$. Thus, we are dealing with a theory with first-class constraints.

3. QUANTIZATION

Let us consider first the Dirac quantization, where the second-class constraints define the Dirac brackets and therefore the commutation relations, whereas the first-class constraints, being applied to the state vectors, define physical states. For essential operators and nonzeroth commutation relations one can obtain in the case under consideration

$$[\hat{x}^\mu, \hat{\pi}_\nu] = \iota\{x^\mu, \pi_\nu\}_{D(\Phi_4^{(1)})} = \iota\delta_\nu^\mu, \qquad [\hat{\psi}^a, \hat{\psi}^b]_+ = \iota\{\psi^a, \psi^b\}_{D(\Phi_4^{(1)})} = -\frac{1}{2}\eta^{ab} \tag{11}$$

It is possible to construct a realization of the commutation relations (11) in a Hilbert space $\mathfrak{R}$ whose elements $\mathbf{f} \in \mathfrak{R}$ are 2^{n+1}-component columns dependent on x,

$$\mathbf{f}(x) = \begin{pmatrix} u_-(x) \\ u_+(x) \end{pmatrix} \tag{12}$$

where $u_\mp(x)$ are 2^n-component columns. Then

$$\hat{x}^\mu = x^\mu \mathbf{I}, \qquad \hat{\pi}_\mu = -\iota\partial_\mu \mathbf{I}, \qquad \hat{\psi}^a = \frac{\iota}{2}\gamma^a \tag{13}$$

Here $\mathbf{I}$ is $2^{n+1} \times 2^{n+1}$ unit matrix and γ^a, $a = 0, 1, \ldots, 2n + 1$, are γ-matrices in $2(n + 1)$ dimensions (Case, 1955), which we select in the spinor representation $\gamma^0 = \text{antidiag}(I, I)$, $\gamma^i = \text{antidiag}(\sigma^i, -\sigma^i)$, $i = 1, 2, \ldots, 2n + 1$, where I is a $2^n \times 2^n$ unit matrix, and σ^i are $2^n \times 2^n$ σ-matrixes which obey the Clifford algebra, $[\sigma^i, \sigma^j]_+ = 2\delta^{ij}$.

According to the scheme of quantization selected, the operators of the first-class constraints have to be applied to the state vectors to define the physical sector, namely, $\hat{\Phi}^{(2)}\mathbf{f}(x) = 0$, where $\hat{\Phi}^{(2)}$ are operators which correspond to the constraints (10). There is no ambiguity in the construction of the operator $\hat{\Phi}_1^{(2)}$ according to the classical function $\Phi_1^{(2)}$. Taking into account the realization (12), (13), one can represent the equations $\hat{\Phi}^{(2)}\mathbf{f}(x) = 0$ in the 2^n-component form

$$[\iota\partial_\mu\gamma^\mu - m\gamma^{2n+1}]\mathbf{f}(x) = 0 \Leftrightarrow \begin{cases} [\iota\partial_\mu\Gamma_+^\mu - m]u_+(x) = 0 \\ [\iota\partial_\mu\Gamma_-^\mu + m]u_-(x) = 0 \end{cases} \tag{14}$$

where two sets of γ-matrices Γ_ζ^μ, $\zeta = \pm$, in $2n + 1$ dimensions are introduced,

$$\Gamma_\zeta^0 = \sigma^{2n+1}, \qquad \Gamma_\zeta^1 = \zeta\sigma^{2n+1}\sigma^1, \ldots, \qquad \Gamma_\zeta^{2n} = \zeta\sigma^{2n+1}\sigma^{2n}$$

$$\Gamma_-^\mu = \Gamma_{+\mu}, \qquad [\Gamma_\zeta^\mu, \Gamma_\zeta^\nu]_+ = 2\eta^{\mu\nu} \tag{15}$$

There is a relation $\hat{\Phi}_2^{(2)} = (\hat{\Phi}_1^{(2)})^2$, so that the equation $\hat{\Phi}_2^{(2)}\mathbf{f} = 0$ is not independent. The equation $\hat{\Phi}_3^{(2)}\mathbf{f}(x) = 0$ can be represented in the form

$$\left[\frac{(-\iota)^n}{(2n)!}\epsilon^{\mu\rho_1\ldots\rho_{2n}}(\iota\partial_\mu)\gamma_{\rho_1}\ldots\gamma_{\rho_{2n}} + sm\right]\mathbf{f}(x) = 0$$

or in 2^n-component form

$$[\iota\partial_\mu\Gamma_+^\mu + (-1)^n sm]u_+(x) = 0$$

$$[\iota\partial_\mu\Gamma_-^\mu + (-1)^n sm]u_-(x) = 0 \tag{16}$$

In quantum theory one has to select $s = \pm 1$; then, combining equations (14) and (16), we get

$$[\iota\partial_\mu \Gamma_s^\mu - \zeta m]u_\zeta(x) = 0, \qquad u_{-\zeta}(x) \equiv 0, \qquad \zeta = (-1)^n s = \pm 1 \quad (17)$$

To interpret the result obtained one has to calculate also the operators $\hat{M}_{\mu\nu}$ corresponding to the angular momentum tensor (6),

$$\hat{M}_{\mu\nu} = -\iota(x_\mu\partial_\nu - x_\nu\partial_\mu) - \frac{\iota}{4}\begin{pmatrix} [\Gamma_{-\mu}, \Gamma_{-\nu}] & 0 \\ 0 & [\Gamma_{+\mu}, \Gamma_{+\nu}] \end{pmatrix}$$

Thus, in the quantum mechanics constructed, the states with $\zeta = +$ are described by the 2^n-component wave function $u_+(x)$, which obeys the Dirac equation in $2n + 1$ dimensions and is transformed under the Lorentz transformation as spin $+1/2$. For $\zeta = -$ the quantization leads to the theory of the $2n + 1$ Dirac particle with spin $-1/2$ and the wave function $u_-(x)$.

To quantize the theory canonically we have to impose as much as possible supplementary gauge conditions to the first-class constraints. In the case under consideration, it turns out to be possible to impose gauge conditions on all the first-class constraints, excluding the constraint $\tilde{\Phi}_3^{(2)}$. Thus, we fix the gauge freedom which corresponds to two types of gauge transformations (3) and (4). As a result we remain only with one first-class constraint, which is a reduction of $\Phi_3^{(2)}$ to the rest of the constraints and gauge conditions. It can be used to specify the physical states. All the second-class constraints form the Dirac brackets. The following gauge conditions $\Phi^G = 0$ are imposed: $\Phi_1^G = e + \zeta\pi_0^{-1}$, $\Phi_2^G = \chi$, $\Phi_3^G = \kappa$, $\Phi_4^G = x_0 - \zeta\tau$, and $\Phi_5^G = \psi^0$, where $\zeta = -\text{sign}\ \pi^0$. [The gauge $x_0 - \zeta\tau = 0$ was first proposed in Gitman and Tyutin (1990a–c) as a conjugate gauge condition to the constraints $\pi^2 - m^2 = 0$.] Using the consistency condition $\dot{\Phi}^G = 0$, one can determine the Lagrangian multipliers which correspond to the primary constraints $\Phi_i^{(1)}$, $i = 1, 2, 3$. To go over to a time-independent set of constraints [to use the standard scheme of quantization without any modifications (Gitman and Tyutin, 1990b)], we introduce the variable x_0', $x_0' = x_0 - \zeta\tau$, instead of x_0, without changing the rest of the variables. That is a canonical transformation in the space of all variables with the generating function $W = x_0\pi_0' + \tau|\pi_0'| + W_0$, where W_0 is the generating function of the identity transformation with respect to all variables except x^0 and π_0. The transformed Hamiltonian $H^{(1)\prime}$ is of the form

$$H^{(1)\prime} = H^{(1)} + \frac{\partial W}{\partial \tau} = \omega + \{\Phi\},$$

$$\omega = \sqrt{\pi_d^2 + m^2}, \qquad d = 1, 2, \ldots, 2n \quad (18)$$

where $\{\Phi\}$ are terms proportional to the constraints and ω is the physical Hamiltonian. All the constraints of the theory can be represented after this canonical transformation in the following equivalent form: $K = 0$, $\phi = 0$, $T = 0$, where

$$K = (e - \omega^{-1}, P_e; \chi, P_\chi; \kappa, P_\kappa; x'_0, |\pi_0| - \omega; \psi^0, P_0)$$

$$\phi = (\pi_d \psi^d + m\psi^{2n+1}, P_k + \iota\psi_k), \qquad k = 1, 2, \ldots, 2n + 1$$

$$T = \frac{(2\iota)^n}{(2n)!} \zeta\omega\epsilon^{i_1 \ldots i_{2n}} \psi_{i_1} \ldots \psi_{i_{2n}} + \frac{sm}{2^n}, \qquad i_d = 1, 2, \ldots, 2n \tag{19}$$

The constraints K and ϕ are of the second class, whereas T is the first-class constraint. The set K has a so-called special form (Gitman and Tyutin, 1990b). In this case, if we eliminate the variables e, P_e, χ, P_χ, κ, P_κ, x'_0, $|\pi_0|$, ψ^0, and P_0, using the constraint $K = 0$, the Dirac brackets with respect to all the second-class constraints (K, ϕ) reduce to ones with respect to the constraints ϕ only. Thus, at this stage, we will only consider the variables x^d, π_d, ζ, ψ^k, and P_k and two sets of constraints—the second-class ones ϕ and the first-class one T. Nonzeroth Dirac brackets for the independent variables are

$$\{x^d, \pi_r\}_{D(\phi)} = \delta^d_r, \qquad \{x^d, x^r\}_{D(\phi)} = \frac{\iota}{\omega^2} [\psi^d, \psi^r],$$

$$\{x^d, \psi^r\}_{D(\phi)} = -\frac{1}{\omega^2} \psi^d \pi_r$$

$$\{\psi^d, \psi^r\}_{D(\phi)} = -\frac{\iota}{2} (\delta^d_r - \omega^{-2}\pi_d \pi_r), \qquad d, r = 1, 2, \ldots, 2n \tag{20}$$

Going over to the quantum theory, we get the commutation relations between the operators $\hat{x}^d$, $\hat{\pi}_d$, $\hat{\psi}^d$ by means of the Dirac brackets (20),

$$[\hat{x}^d, \hat{\pi}_r] = \iota\delta^d_r, \qquad [\hat{x}^d, \hat{x}^r] = -\frac{1}{\hat{\omega}^2} [\hat{\psi}^d, \hat{\psi}^r]$$

$$[\hat{x}^d, \hat{\psi}^r] = -\frac{\iota}{\hat{\omega}^2} \hat{\psi}^d \hat{\pi}_r, \qquad [\hat{\psi}^d, \hat{\psi}^r]_+ = \frac{1}{2} (\delta^d_r - \hat{\omega}^{-2}\hat{\pi}_d \hat{\pi}_r) \tag{21}$$

We assume as usual (Gitman and Tyutin, 1990a–c) that the operator $\hat{\zeta}$ has the eigenvalues $\zeta = \pm 1$ by analogy with the classical theory, so that $\hat{\zeta}^2 = 1$, and also we assume the equations of the second-class constraints $\hat{\phi} = 0$. Then one can realize the algebra (21) in a Hilbert space $\mathfrak{R}$ whose elements $\mathbf{f} \in \mathfrak{R}$ are 2^{n+1}-component columns dependent on $\mathbf{x} = (x^d)$, $d = 1, 2, \ldots, 2n$,

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_+(\mathbf{x}) \\ f_-(\mathbf{x}) \end{pmatrix} \tag{22}$$

so that $f_+(\mathbf{x})$ and $f_-(\mathbf{x})$ are 2^n-component columns. A realization of the commutations relations has the form

$$\hat{x}^d = x^d\mathbf{I} - \frac{\iota}{4\hat{\omega}^2}[\Sigma^d, \hat{\pi}_r\Sigma^r]_- - \frac{\iota m}{4\hat{\omega}^2}[\Sigma^d, \Sigma^{2n+1}]_-, \qquad \hat{\pi}_r = -\iota\partial_r\mathbf{I}$$

$$\hat{\psi}^d = \frac{1}{2}(\delta_r^d - \hat{\omega}^{-2}\hat{\pi}_d\hat{\pi}_r)\Sigma^r - \frac{m\hat{\pi}_d}{2\hat{\omega}^2}\Sigma^{2n+1}, \qquad \hat{\zeta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{23}$$

where $\mathbf{I}$ and I are $2^{n+1} \times 2^{n+1}$ and $2^n \times 2^n$ unit matrices, and $\Sigma^k = \mathrm{diag}(\sigma^k, \sigma^k)$. The operator $\hat{T}$ corresponding to the first-class constraint T [see (19)] is

$$\hat{T} = \frac{\zeta m}{\hat{\omega}}\hat{\zeta}\Sigma^{2n+1}[\hat{\zeta}\hat{\omega}\Sigma^{2n+1} + \iota\partial_d(\zeta\Sigma^{2n+1}\Sigma^d) - \zeta m], \qquad \zeta = (-1)^n s = \pm 1 \tag{24}$$

The latter operator specifies the physical states according to the scheme of quantization selected, $\hat{T}\mathbf{f} = 0$. On the other hand, the state vectors $\mathbf{f}$ have to obey the Schrödinger equation, which defines their "time" dependence, $(\iota\partial/\partial\tau - \hat{\omega})\mathbf{f} = 0$, $\hat{\omega} = (\hat{\pi}_d^2 + m^2)^{1/2}$, where the quantum Hamiltonian $\hat{\omega}$ corresponds the classical one ω, (18). Introducing the physical time $x^0 = \zeta\tau$ instead of the parameter τ (Gitman and Tyutin, 1990a–c), we can rewrite the Schrödinger equation in the following form [we can now write $\mathbf{f} = \mathbf{f}(x)$, $(x = x^0, \mathbf{x})$]:

$$\left(\iota\frac{\partial}{\partial x^0} - \hat{\zeta}\hat{\omega}\right)\mathbf{f}(x) = 0 \tag{25}$$

Using (25) in the equation $\hat{T}\mathbf{f} = 0$, namely replacing there the combination $\hat{\zeta}\hat{\omega}\mathbf{f}$ by $\iota\partial_0\mathbf{f}$, one can verify that both components $f_\pm(x)$ of the state vector (22) obey one and the same equation

$$(\iota\partial_\mu\Gamma_\zeta^\mu - \zeta m)f_\zeta(x) = 0, \qquad \zeta = \pm 1 \tag{26}$$

which is the $2n + 1$ Dirac equation for a particle of spin $\zeta/2$, whereas $f_\pm(x)$ can be interpreted [taking into account (25)] as the positive- and negative-frequency solutions to the equation, respectively. Substituting the realization (23) into the expression (6), we get the generators of the Lorentz transformations

$$\hat{M}_{\mu\nu} = -\iota(x_\mu\partial_\nu - x_\nu\partial_\mu) - \frac{\iota}{4}\begin{pmatrix} [\Gamma_{\zeta\mu}, \Gamma_{\zeta\nu}] & 0 \\ 0 & [\Gamma_{\zeta\mu}, \Gamma_{\zeta\nu}] \end{pmatrix} \tag{27}$$

which have the standard form for both components $f_\zeta(x)$. Thus, a natural interpretation of the components $f_\zeta(x)$ is the following: $f_+(x)$ is the wave function of a particle with spin $\zeta/2$ and $f_-^*(x)$ is the wave function of an antiparticle with spin $\zeta/2$.

4. DIMENSIONAL DUALITY BETWEEN MASSIVE AND MASSLESS SPINNING PARTICLES

As is known, the method of dimensional reduction (Duff *et al.*, 1986; Green *et al.*, 1988) is often useful to construct models (actions) in low dimensions using appropriate models in higher dimensions. In fact, ideas began from the work of Kaluza (1921) and Klein (1926). One can also mention that the method of dimensional reduction was used to interpret masses in supersymmetric theories as components of momenta in space of higher dimensions, which are frozen in the course of the reduction. It is interesting that the model of Dirac particles in odd dimensions proposed in the present paper is related to the model (Gitman and Gonçalves, 1995; Grigoryan *et al.*, 1995) of Weyl particles in even dimensions by means of a dimensional reduction.

The action and the Hamiltonian of the latter model in $D = 2(n + 1)$ dimensions have the form

$$S = \int_0^1 \left[-\frac{z^2}{2e} - \iota\psi_\mu\dot{\psi}^\mu \right] d\tau$$

$$z^\mu = \dot{x}^\mu - \iota\psi^\mu\chi + \frac{(2\iota)^{(D-2)/2}}{(D-2)!}\epsilon^{\mu\nu\rho_2\cdots\rho_{D-1}}b_\nu\psi_{\rho_2}\ldots\psi_{\rho_{D-1}} + \frac{s}{2^{(D-2)/2}}b^\mu$$

$$H = -\frac{e}{2}\pi^2 + \iota\pi_\mu\psi^\mu\chi - \left[\frac{(2\iota)^{(D-2)/2}}{(D-2)!}\epsilon_{\nu\mu\rho_2\cdots\rho_{D-1}}\pi^\mu\psi^{\rho_2}\ldots\psi^{\rho_{D-1}} + \frac{\alpha}{2^{(D-2)/2}}\pi_\nu\right]b^\nu \quad (28)$$

In the canonical gauge similar to one which was considered above, in particular, in the gauge $\psi^0 = 0$, only the first-class constraints remain,

$$T_\mu = \frac{(2\iota)^{(D-2)/2}}{(D-2)!}\epsilon_{\nu\mu\rho_2\cdots\rho_{D-1}}\pi^\mu\psi^{\rho_2}\ldots\psi^{\rho_{D-1}} + \frac{\alpha}{2^{(D-2)/2}}\pi_\nu = 0 \quad (29)$$

One can see that, in fact, among the constraints (29) only one is independent,

$$T_\mu = \frac{\pi_\mu}{\pi_0}T_0$$

Thus one can use only one component of b^μ and put all the others to zero. Now one can do a dimensional reduction $2(n + 1) \to 2n + 1$ in the Hamiltonian and constraint T_0, putting also $\pi_{2n+1} = m$, $b^{2n+1} = -\kappa$, $b^0 = b^1 = \ldots = b^{2n} = 0$. As a result of such a procedure we just obtain the expressions (9) and (19) for the Hamiltonian and the constraint. The second-class constraints of the model (28) also coincide with those of the model (2) after the dimensional reduction. Thus, there exists a dimensional duality between the massive spinning particles in odd dimensions and massless ones in even dimensions.

5. DISCUSSION

There are pseudoclassical models (PM) to describe all massive higher spins (integer and half-integer) in 3 + 1 dimensions (Berezin and Marinov, 1975, 1977; Brink *et al.*, 1976, 1977; Casalbuoni, 1976; Barducci *et al.*, 1976; Balachandran *et al.*, 1977; Henneaux and Teitelboim, 1982; Gitman and Tyutin, 1990a; Fradkin and Gitman, 1991; Srivastava, 1977; Gershun and Tkach, 1979; Howe *et al.*, 1988, 1989; Barducci *et al.*, 1976; Marnelius and Martenson, 1990, 1991). Generalization of the models to arbitrary even dimensions can be easily done by means of a trivial dimensional extension similar to the spin-one-half case. To get the PM for higher spins in arbitrary odd dimensions one can start from the model proposed in the present paper, using the ideas of Gitman and Tyutin (1996). Namely, one has to multiply the variables ψ, χ, κ, s in the action (2). Then an appropriate action has the form

$$
S = \int_0^1 \left\{ -\frac{z^2}{2e} - e\,\frac{m^2}{2} - \sum_{A=1}^{N} \left[sm\left(\frac{\kappa_A}{2^n} + i\psi_A^{2n+1}\chi_A \right) + i\psi_{Aa}\dot{\psi}_A^a \right] \right\} d\tau
$$

$$
z^\mu = \dot{x}^\mu - \sum_{A=1}^{N} \left[\iota\psi_A^\mu \chi_A - \frac{(2\iota)^n}{(2n)!}\, \epsilon^{\mu\rho_1\ldots\rho_{2n}} \psi_{A\rho_1} \ldots \psi_{A\rho_{2n}} \kappa_A \right] \qquad (30)
$$

Certainly, a detailed analysis of the action (30) and its quantization may demand significant technical work in higher dimensions. While in 2 + 1 dimensions the model can be quantized explicitly for all higher spins both canonically and by means of the Dirac method (Gitman and Tyutin, 1996), in 3 + 1 dimensions the corresponding PM (Srivastava, 1977; Gershun and Tkach, 1979; Howe *et al.*, 1988, 1989; Barducci *et al.*, 1976; Marnelius and Martenson, 1990, 1991) was quantized canonically only for spins one-half (Berezin and Marinov, 1975, 1977; Brink *et al.*, 1976, 1977; Casalbuoni, 1976; Barducci *et al.*, 1976; Balachandran *et al.*, 1977; Henneaux and Teitelboim, 1982; Gitman and Tyutin, 1990a; Fradkin and Gitman, 1991) and one (Gitman *et al.*, 1995). As to the massless spin-one-half particles, the corresponding pseudoclassical model exists at present in arbitrary even dimensions (Gitman and Gonçalves, 1995; Grigoryan *et al.*, 1995; Gitman *et al.*, 1994b). Its generalization to describe higher spins can be done in the same manner:

$$S = \int_0^1 \left[-\frac{1}{2e} z^2 - \iota \sum_{A=1}^{N} \psi_{A\mu} \dot{\psi}_A^{\mu} \right] d\tau$$

$$z^{\mu} = \dot{x}^{\mu} - \sum_{A=1}^{N} \left(\iota \psi_A^{\mu} \chi_A - \iota \epsilon^{\mu\nu\rho\zeta} b_{A\nu} \psi_{A\rho} \psi_{A\zeta} - \frac{1}{2} s b_A^{\mu} \right) \quad (31)$$

There exists the dimensional duality mentioned above between the models (30) and (31).

Massless higher spins in arbitrary odd dimensions can be described pseudoclassically by the model which follows from (30) in the limit $m \to 0$.

Thus, at present, in principle, we have pseudoclassical models to describe all integer and half-integer spins in arbitrary dimensions.

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